

# Feynman rules and $\beta$ -function for the BF Yang-Mills theory

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## Abstract

Yang-Mills theory in the first order formalism appears as the deformation of a topological field theory, the pure BF theory. We discuss this formulation at the quantum level, giving the Feynman rules of the BF-YM theory, the structure of the renormalization and checking its  $uv$ -behaviour in the computation of the  $\beta$ -function which agrees with the expected result.

# 1 Introduction

Gauge theories, which play a central role in our understanding of high energy interactions, are usually described in terms of the Yang-Mills action. In this letter we consider the first order formulation of Yang-Mills theory, in which an auxiliary tensor field  $B$  couples to the physical degrees of freedom of the gauge theory. This formulation, which has been used in [1, 2] to introduce an explicit representation of the 't Hooft algebra [3], makes closer the connection between Yang-Mills theory and topological field theories of BF type [4, 5]; we will call this formulation BFYM theory. We give Feynman rules for BFYM theory and discuss the structure of one loop divergent diagrams and renormalization and check the  $uv$ -behaviour of the theory computing the  $\beta$ -function which turns out to agree with the expected value; some of these results have been anticipated in [2].

The first order form of pure euclidean Yang-Mills theory is described by the action functional

$$\begin{aligned} S_{BFYM} &= \int \text{Tr}[iB \wedge F + g^2 B \wedge *B] \\ &= \int d^4x \left( \frac{i}{2} \varepsilon^{\mu\nu\alpha\beta} B_{\mu\nu}^a F_{\alpha\beta}^a + g^2 B_{\mu\nu}^a B^{a\mu\nu} \right) , \end{aligned} \quad (1.1)$$

where  $F = F_{\mu\nu}^a dx^\mu \wedge dx^\nu \hat{T}^a$  is the usual field strength,  $D \equiv d + i[A, \cdot]$  and  $B$  is a Lie valued 2-form [6]. The generators of the  $SU(N)$  Lie algebra in the fundamental representation are normalized as  $\text{Tr} \hat{T}^a \hat{T}^b = 1/2 \delta_{ab}$  and  $*$  is the Hodge product for a  $p$ -form. The field equations of (1.1) are

$$\begin{aligned} F &= 2ig^2 * B , \\ DB &= 0 . \end{aligned} \quad (1.2)$$

The standard YM action is recovered performing path integration over  $B$  or by using equations (1.2) in (1.1). Therefore the BFYM action (1.1) is on-shell equivalent to YM theory and its classical gauge invariance is given by

$$\begin{aligned} \delta A &= D\Lambda_0 , \\ \delta B &= i[\Lambda_0, B] . \end{aligned} \quad (1.3)$$

The question arises whether the two formulations are equivalent at the quantum level and to which extent this equivalence holds. Note that off-shell the field  $B$  is not constrained by any Bianchi identity and this fact has been related to the presence of magnetic vortex lines in the vacuum of the theory in [1], in the picture of the dual superconductor vacuum [7, 8].

In the limit of vanishing coupling,  $g \rightarrow 0$ , the action (1.1) flows in the pure BF theory [4, 5] which is known to give a topological field theory,

$$S_{BF} = i \int \text{Tr}[B \wedge F] \quad . \quad (1.4)$$

Indeed the action (1.4) has a second gauge symmetry, namely

$$\begin{aligned} \tilde{\delta} A &= 0 \quad , \\ \tilde{\delta} B &= D\Lambda_1 \quad , \end{aligned} \quad (1.5)$$

where  $\Lambda_1$  is a 1-form. The presence of this “topological” symmetry cancels out any local degree of freedom from the theory (1.4).

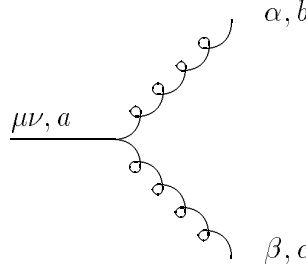
Therefore YM theory in the BF formulation appears as a deformation of the topological field theory (1.4); the  $g^2 B^2$  term which allows gaussian integration in (1.1) is an explicit breaking term for the symmetry (1.5). Since pure BF theory is known to be a finite theory [5], the explicit symmetry breaking is expected to lead to a renormalizable one.

One can cast a perturbative framework in BFYM, in order to check its  $uv$ -behaviour and in comparison with the standard perturbative expansion in YM. Actually there are two different ways to quantize BFYM theory and define Feynman rules. The first one is to regard the topological symmetry breaking term  $g^2 B^2$  belonging to the kinetic part of the lagrangian (1.1); in this way only the gauge symmetry needs gauge fixing and quantization and this is the case considered in this letter. The second one regards the term  $g^2 B^2$  as a true vertex; in this case the kinetic kernel is the same of the pure BF theory and requires gauge fixing and quantization also of the topological symmetry, although anomalous at the classical level. The procedure of quantization of the topological symmetry is quite involved, requiring a ghosts of ghosts structure due to the reducible



appendix, where the relative Ward identities are considered. Note the mass dimensions of the given propagators which accord with the canonical scale dimensions of the fields; the propagator  $\Delta_{BB}$  has dimension zero and behaves as a contact term at high momentum.

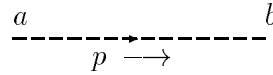
From (1.7) we see that BFYM has no self couplings for the gauge field  $A$ , the only relevant coupling being given by the vertex  $BAA$ :

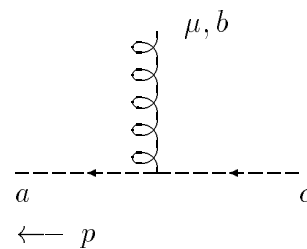


$$\Lambda_{(BAA)\mu\nu\alpha\beta}^{abc} = -i g f^{abc} \varepsilon_{\mu\nu\alpha\beta} \quad .$$

**Fig. 2**

As we will see is the off-diagonal structure of the propagators which reproduces the non linear self couplings of the gauge field. To these Feynman rules we must add the usual ghost ones



$$\Delta_{c\bar{c}}^{ab}(p) = -\frac{i}{p^2} \delta^{ab} \quad ,$$


$$\Lambda_{(Ac\bar{c})\mu}^{abc}(p) = g p_\mu f^{abc} \quad .$$

**Fig. 3**

## 2 One Loop two point functions

In this section we consider the calculation of one-loop self-energies. In Fig. 4 are shown all the relevant one loop diagrams. The calculations are done in the Landau gauge ( $\alpha = 0$ ) using dimensional regularization, in dimension  $D = 4 - 2\epsilon$ . In order to study the one loop

renormalization and the  $\beta$ -function of BFYM theory we consider only the divergent parts of the diagrams considered.

The regularized divergent part of the self energies are then given by

$$\Pi_{AA} = \frac{1}{6}g^2z(\epsilon)\delta^{ab}(p^2\delta^{\mu\nu} - p^\mu p^\nu) \quad (2.1)$$

for the gluon self energy; by

$$\Pi_{AB} = \frac{3}{4}g^2z(\epsilon)\delta^{ab}\varepsilon^{\alpha\beta\lambda\nu}p^\lambda \quad (2.2)$$

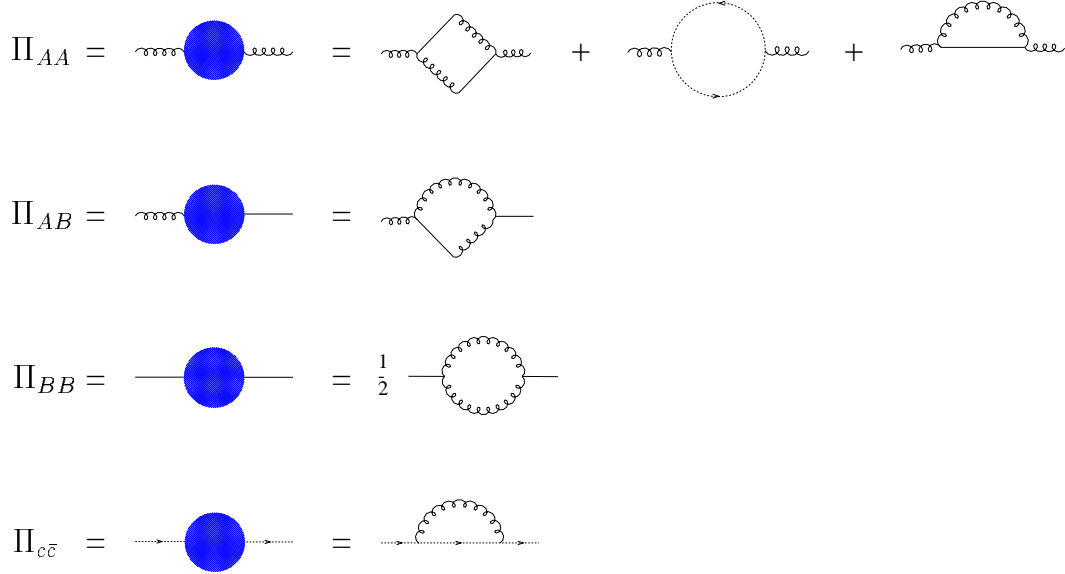
for the  $AB$  self energy; by

$$\Pi_{BB} = -\frac{1}{2}g^2z(\epsilon)\delta^{ab}I_{[\mu\nu][\alpha\beta]} \quad (2.3)$$

for the  $B$  self energy and by

$$\Pi_{c\bar{c}} = \frac{3}{4}g^2z(\epsilon)\delta^{ab}p^2 \quad (2.4)$$

for the ghost self energy.  $z(\epsilon) \equiv c_V \frac{\Gamma(\epsilon)}{(4\pi)^2}$ , where  $c_V$  is the quadratic casimir for the Lie algebra of  $SU(N)$ .  $I_{[\mu\nu][\alpha\beta]} = \delta^{\mu\alpha}\delta^{\nu\beta} - \delta^{\mu\beta}\delta^{\nu\alpha}$  is the antisymmetric identity.



**Fig. 4**

As far as the self energies involving the  $A$  and  $B$  fields are concerned, note that they all contribute to the one loop two point functions. For example consider the correlator  $\langle AA \rangle$ ; in Fig. 5 we see how due to the structure of the propagator matrix the one loop contribution to this function is recovered.

$$\begin{aligned}
\langle A(x)A(y) \rangle = & \quad x \text{---} \text{wavy} \text{---} \text{blue circle} \text{---} \text{wavy} \text{---} y + x \text{---} \text{wavy} \text{---} \text{blue circle} \text{---} \text{wavy} \text{---} y \\
& + x \text{---} \text{wavy} \text{---} \text{blue circle} \text{---} \text{wavy} \text{---} y + x \text{---} \text{wavy} \text{---} \text{blue circle} \text{---} \text{wavy} \text{---} y
\end{aligned}$$

**Fig. 5**

In an analogous fashion it works for the two point functions  $\langle AB \rangle$  and  $\langle BB \rangle$ .

### 3 Vertex One Loop Diagrams

The superficial degree of divergence for the vertex diagrams is given by the formula

$$\omega = 4 - (E_A + E_c) - 2E_B \quad , \quad (3.1)$$

where  $E_A$ ,  $E_B$  and  $E_c$  represent the number of external legs joined to the diagram *via*  $A$ ,  $B$  and  $c$  respectively. We then obtain for BFYM theory the four divergent vertex diagrams reported in Fig. 6.

The full calculation of these vertices should take into account more over sixty diagrams including permutations; we restrict the calculation only to the divergent parts of the first two vertices. The divergent part of the ghost vertex  $\Gamma_{Ac\bar{c}}$  is vanishing as in the standard calculations, owing to the transversality of the propagators in the Landau gauge. For the same reasons also the vertex  $\Gamma_{BAA}$  is found to be finite at one loop order; this is the same vertex of the pure BF theory and seems to behave in a fashion corresponding to the topological theory. The last two vertices,  $\Gamma_{AAA}$  and  $\Gamma_{AAAA}$  do not belong to the tree level BFYM action and correspond to the nonlinear self interactions of YM which in this way are recovered into the theory. These vertices, joined to the gluon self energy (2.1), originate from an  $F^2$  term which the symmetries of the theory allow to enter in the quantum action; we will see in the next section how renormalization has to be performed in order to produce all the required counterterms.

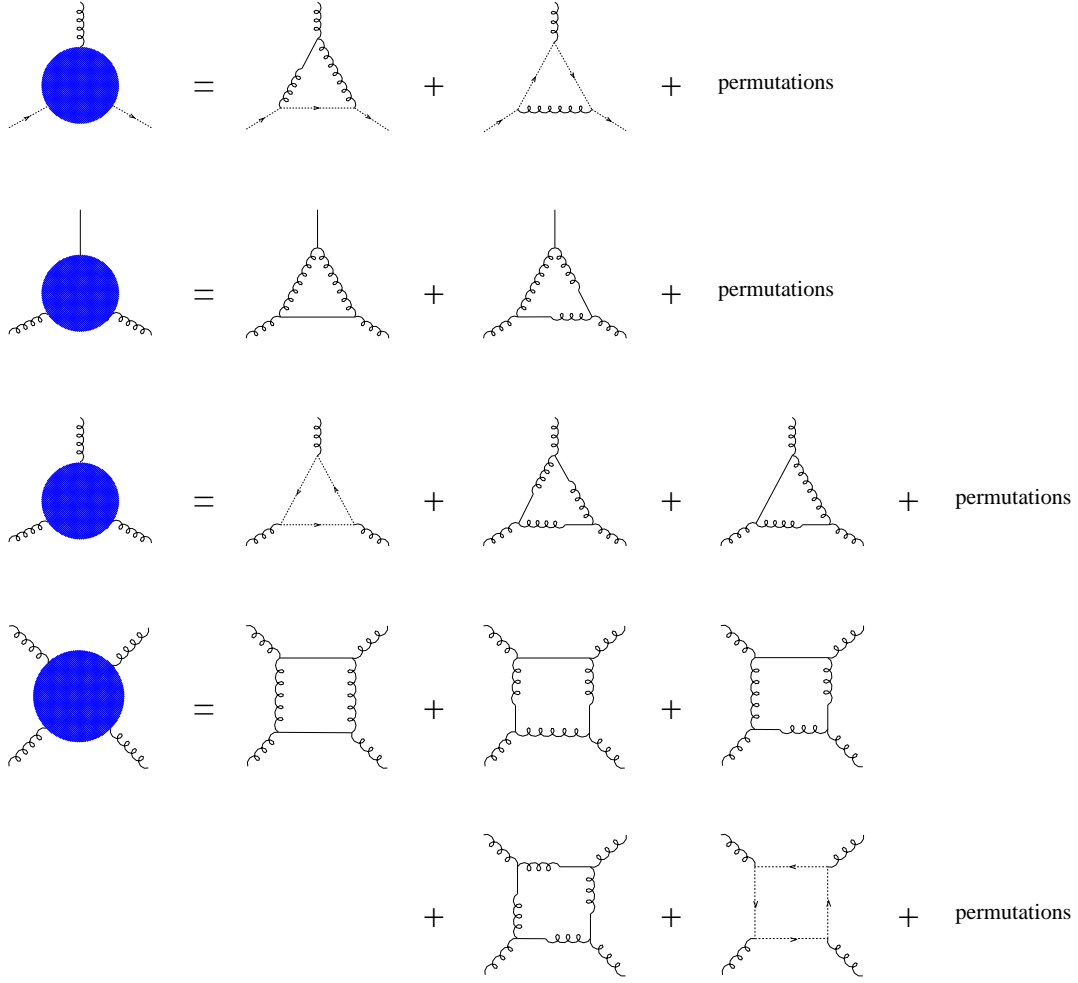


Fig. 6

## 4 Renormalization and $\beta$ -function

Renormalization is performed substituting the bare quantities with the renormalized ones, where in general an operatorial mixing is allowed by the symmetry and parity properties of the fields. We then write

$$\begin{aligned} B_0 &= Z_{BB}B_R + iZ_{BA} * F_R \ , \\ F_0 &= Z_{AA}F_R \ , \end{aligned} \tag{4.1}$$

with  $F_R = dA_R + Z_{AA}Z_g g_R[A_R, A_R]$ , where  $g_0 = Z_g g_R$  and where the gauge Ward identities among renormalization constants have been imposed. Note that  $B$  and  $F$  have opposite parity; moreover a mixing of  $B_R$  in  $F_0$  is not allowed since  $F$  must be a curvature tensor.



This field mixing introduces the term  $F^2$  absent at tree level in the theory and the counterterms relative to the gluon self-energy and to trilinear and quadrilinear gluon vertices. We obtain

$$\begin{aligned}
S_{BFYM} &= \int \text{Tr}[iB_0 \wedge F_0 + B_0 \wedge *B_0] \\
&= \int \text{Tr}[iZ_{BB}(Z_{AA} + 2Z_{BA})B_R \wedge F_R + Z_{BB}^2 B_R \wedge *B_R \\
&\quad - Z_{BA}(Z_{BA} + Z_{AA})F_R \wedge *F_R] \quad .
\end{aligned} \tag{4.2}$$

The renormalization of the ghost terms is performed in the usual way. Since Feynman rules of BFYM at tree level should not be modified we expect

$$\begin{aligned}
Z_{AA} &\simeq 1 + ag_R^2 z(\epsilon) + (...) + o(g_R^2) \quad , \\
Z_{BB} &\simeq 1 + bg_R^2 z(\epsilon) + (...) + o(g_R^2) \quad , \\
Z_{BA} &\simeq cg_R^2 z(\epsilon) + (...) + o(g_R^2) \quad ,
\end{aligned} \tag{4.3}$$

where dots (...) represent finite terms at order  $g_R^2$ . The value of the elements of the wave function renormalization matrix are assigned by direct comparison between the Feynman rules for the quadratic counterterms (4.2) and the divergent parts of the self-energies (2.1-2.4). We obtain the following system

$$\begin{aligned}
Z_{BB}(Z_{AA} + 2Z_{BA}) &= 1 + \frac{3}{4}g_R^2 z(\epsilon) + (...) + o(g_R^2) \quad , \\
Z_{BB}^2 &= 1 - \frac{1}{2}g_R^2 z(\epsilon) + (...) + o(g_R^2) \quad , \\
4(Z_{BA}^2 + Z_{BA}Z_{AA}) &= -\frac{1}{6}g_R^2 z(\epsilon) + (...) + o(g_R^2) \quad ;
\end{aligned} \tag{4.4}$$

(note that the factor 4 in the third equation is due to the usual normalization for the  $F^2$  term). Solving the equations (4.4) at the order  $g^2$  we find

$$a = \frac{13}{12} \quad , \quad b = -\frac{1}{4} \quad , \quad c = -\frac{1}{24} \quad . \tag{4.5}$$

The value of  $a$  gives exactly the wave function renormalization for  $A$  required in the Landau gauge for obtaining the correct value for the  $\beta$ -function of the theory. Indeed, introducing the ghost wave function renormalization,  $c_0 = Z_c c_R$ , from (2.4) we read

$$Z_c = 1 + \frac{3}{8}z(\epsilon)g_R^2 + (...) + o(g_R^2) \quad , \tag{4.6}$$

and from the finiteness of the gluon-ghost vertex in Landau gauge we obtain

$$Z_g Z_{AA} Z_c^2 = 1 + (...) + o(g_R^2) \quad , \quad (4.7)$$

where no divergent part at order  $g_R^2$  is present. From (4.7) the renormalization of the coupling constant turns out to be

$$Z_g = 1 - \frac{11}{6} z(\epsilon) g_R^2 + (...) + o(g_R^2) \quad , \quad (4.8)$$

which gives  $\beta_1 = -\frac{11}{3}$  [10]. Therefore, as expected, the  $uv$ -behaviour of BFYM is the same of YM. Also note that the values found in (4.5) give for the divergent part of the  $BAA$  counterterm at  $g_R^2$  level

$$Z_{BB}(Z_{AA} + 2Z_{BA})Z_{AA}Z_g = 1 + (...) + o(g_R^2) \quad , \quad (4.9)$$

according to the finiteness of  $\Gamma_{BAA}$ . After renormalization is performed is always possible to redefine  $B_R$  in order to reabsorb the  $F^2$  term and recover the tree level structure of the theory. Indeed defining

$$\begin{aligned} \tilde{B}_R &= B_R + i\xi * F_R \quad , \\ \tilde{F}_R &= F_R \quad , \end{aligned} \quad (4.10)$$

where at  $g_R^2$  order  $\xi = Z_{BA}/Z_{BB}$ , the renormalized action (4.2) becomes

$$S = \int \text{Tr}[iZ_{BB}Z_{AA}\tilde{B}_R \wedge \tilde{F}_R + Z_{BB}^2 \tilde{B}_R \wedge * \tilde{B}_R] \quad . \quad (4.11)$$

The transformation (4.10) gives a finite renormalization and, not involving the coupling  $g_R$  contained in  $F_R$ , does not modify the correspondence with the renormalized Yang-Mills theory

$$S = \frac{1}{4} \int \text{Tr}[Z_{AA}^2 F_R \wedge * F_R] \quad . \quad (4.12)$$

In conclusion we have shown that this theory can be given a proper perturbative expansion and that the asymptotic free behaviour of BFYM coincides with that of YM. The perturbative formulation and the study of renormalization can be further investigated using algebraic and cohomological tools and indeed the 3D BFYM theory has been studied

in this way [11] and the analysis will be extended to the 4D case. Some perturbative work on BF-type formulation of gravity theories can be found also in [12].

BFYM formulation opens the study to the relations between BF and gauge theories; in particular new non local observables can be introduced. These observables, describing topological higher linking numbers, were introduced in BF theories in [13] and can be naturally introduced in the gauge theory using the enlarged field content of BFYM. This investigation, discussed in [1, 2] and previously started in [14, 15], should be even more richer in the non minimal formulation [9] where the whole content of topological fields is present, added with new vectorial degrees of freedom, and is at most promising to produce a deeper understanding of the non perturbative sector of gauge theories.

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## A Appendix

In order to compute propagators consider  $S_0 + \int J\Phi$ , the quadratic part of the action (1.1) with the coupling of the fields to the external sources. Propagators are easily derived shifting for example the fields in momentum space by means of field independent functions,  $A(p), B(p), b(p) \rightarrow A(p) + C_A(p), B(p) + C_B(p), b(p) + C_b(p)$ , and solving for the  $C$ 's in such a way that linear terms in the fields disappear [16].  $b$  is the auxiliary field which implements the gauge fixing condition and playing a role only in the inversion of the kinetic operator. The corresponding solution is given by

$$C_A(p)_\mu^a = -\frac{1}{p^4}(p^2 J_{A_\mu}^a - p_\mu p^\nu J_{A_\nu}^a) - \frac{1}{2p^2} \varepsilon^{\alpha\beta\nu\mu} p_\nu J_{B_{\alpha\beta}}^a - i \frac{p_\mu}{p^2} J_b^a \quad ,$$

$$\begin{aligned}
C_B(p)_{\mu\nu}^a &= \frac{1}{2p^2}(p_\mu p^\alpha J_{B\nu\alpha}^a + p_\nu p^\alpha J_{B\alpha\mu}^a) + \frac{1}{2p^2}\varepsilon^{\mu\nu\alpha\beta}p_\alpha J_{A\beta}^a \quad , \\
C_b(p)^a &= \frac{i}{p^2}p^\alpha J_{A\alpha}^a \quad ,
\end{aligned}$$

and produces the propagators given in Fig. 1 with the exception of the  $\Delta_{BB}$  term, which turns out to be

$$\tilde{\Delta}_{BB\mu\nu\alpha\beta}^{ab} = \frac{\delta^{ab}}{4p^2}(p^\nu p^\beta \delta^{\alpha\mu} + p^\alpha p^\mu \delta^{\nu\beta} - p^\nu p^\alpha \delta^{\mu\beta} - p^\mu p^\beta \delta^{\alpha\nu}) \quad . \quad (\text{A.1})$$

Note that this propagator is not transversal. Indeed one loop calculations of the self energy  $\Delta_{BB}$  show that the correct structure is that reported in Fig. 1 and this fact agrees with what predicted by the Ward identity

$$\partial_\mu \partial_\alpha \Delta_{BB\mu\nu\alpha\beta} = 0 \quad , \quad (\text{A.2})$$

which can be derived by differentiation of the Ward identity on the connected Green functions generator functional.

To understand the mismatch between (A.1) and  $\Delta_{BB}$  note that in our treatment we have left undetermined the measure over  $B$  using the naive one. Indeed the correspondence between first and second order formalism should be written as

$$\int [\mathcal{D}B][\mathcal{D}A]e^{-S_{BFYM}} \simeq \int [\mathcal{D}A]e^{-S_{YM}} \quad , \quad (\text{A.3})$$

where  $[\mathcal{D}A]$  is the usual gauge fixed measure and  $[\mathcal{D}B]$  is the measure over the orbits of the topological group (1.5). In our measure instead we have also the integration over the zero modes of the topological group, i.e. the configurations  $B$  such that  $B = D\eta$ ,  $\eta$  1-form, which are not coupled to  $F$  and do not contribute to the gaussian integration owing to the Bianchi identity. They give the overall factor

$$\hat{\Delta}_{BB\mu\nu\alpha\beta}^{ab} = \frac{\delta^{ab}}{4}I_{[\mu\nu][\alpha\beta]} \quad . \quad (\text{A.4})$$

This contact term is exactly the amount of the mismatch found,  $\Delta = \tilde{\Delta} - \hat{\Delta}$ . Therefore we have to take into account the presence of the spurious contribution of topological zero modes and assign to  $\Delta_{BB}$  the correct tensorial structure following the Ward identity.

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